Name	Date started	Target end date

GCE A LEVEL — FURTHER PURE MATHEMATICS B QUESTION PACK

0979-01 (Legacy FP3) · New spec A2 Unit 4 Topic 7

REVISE
.wales

FURTHER MATHS — FP B · HYPERBOLIC FUNCTIONS

Hyperbolic Functions — Definitions, Identities & Inverse Log Forms

Every hyperbolic-functions question from the legacy WJEC FP3 papers (June 2006 – June 2017 + Specimen) that maps onto the new-spec A2 Unit 4 Topic 7.

LEGACY 2008 SPECIFICATION

Estimated time for entire question pack: ~3 hours 3 minutes

*Derived from the legacy FP3 paper's pace of **~1.5 min/mark** (122 marks over 15 questions). The full Unit 4 exam is **2 hours 30 minutes for 120 marks** (35% of the A-level qualification).*

*You are advised to **not** attempt to complete all of this in one sitting.*

ABOUT THIS QUESTION PACK

This is a **comprehensive practice question pack**, not a single mock paper. It contains every hyperbolic functions question from the legacy WJEC FP3 papers (2008 modular spec) that maps onto new-spec A2 Unit 4 Topic 7 (2.4.7). Unit 4 (Further Pure Mathematics B) is the **120-mark compulsory A2 paper**, worth 35% of the A-level qualification.

Questions are ordered roughly by topic / difficulty.

INSTRUCTIONS

Use black ink or black ball-point pen. Show all working — method marks are awarded for clear setup.

A calculator is allowed. The WJEC Formula Booklet may be referred to.

For Examiner's use only

Q	Source	Max	Mark
1	Jun 06 Q1	9	
2	Jun 07 Q1	8	
3	Jun 08 Q2	8	
4	Jun 08 Q4	12	
5	Jun 09 Q1	7	
6	Jun 09 Q3	8	
7	Jun 10 Q2	7	
8	Jun 11 Q1	8	

Q	Source	Max	Mark
9	Jun 12 Q2	7	
10	Jun 13 Q1	6	
11	Jun 14 Q1	9	
12	Jun 14 Q7	11	
13	Jun 15 Q1	8	
14	Jun 17 Q1	7	
15	Spec. Q1	7	
	Total	122	

Hyperbolic Functions — Definitions, Identities & Inverse Log Forms — what the new spec asks

WJEC GCE A Level Further Mathematics (from 2017) · Unit 4: Further Pure Mathematics B · Topic 2.4.7.

Definitions from e^x 2.4.7

- $\sinh x = \dfrac{e^x - e^{-x}}{2}$ — odd function, range $\mathbb{R}$.
- $\cosh x = \dfrac{e^x + e^{-x}}{2}$ — even function, range $[1, \infty)$.
- $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ — range $(-1, 1)$.
- Pythagorean: $\cosh^2 x - \sinh^2 x = 1$.

Identities & calculus 2.4.7

- $\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B$.
- $\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$.
- $\sinh 2A = 2 \sinh A \cosh A$; $\cosh 2A = \cosh^2 A + \sinh^2 A = 2\cosh^2 A - 1$.
- Derivatives: $\frac{d}{dx} \sinh x = \cosh x$, $\frac{d}{dx} \cosh x = \sinh x$, $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$.

Inverse hyperbolics 2.4.7

- $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$ — all x.
- $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$ — $x \geq 1$.
- $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$ — $|x| < 1$.
- Integrals: $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1} \frac{x}{a} + C$ and $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1} \frac{x}{a} + C$.

Working scientifically general

- For equations of form $a \cosh x + b \sinh x = c$, replace with e^x and solve quadratic in e^x.
- For substitution $x = a \sinh u$: $dx = a \cosh u \, du$ and $\sqrt{a^2+x^2} = a \cosh u$.
- For $x = a \cosh u$: $dx = a \sinh u \, du$ and $\sqrt{x^2-a^2} = a \sinh u$.
- Always state the domain restriction on $\cosh^{-1}$ $(x \geq 1)$ — lose marks if you don't.

Hyperbolic Functions in one page

Quick-reference notes — revisit before each section. Don't use during questions.

Definitions

$\sinh x = \dfrac{e^x - e^{-x}}{2}$ — odd; range $\mathbb{R}$.

$\cosh x = \dfrac{e^x + e^{-x}}{2}$ — even; range $[1, \infty)$.

$\tanh x = \dfrac{\sinh x}{\cosh x}$ — odd; range $(-1, 1)$.

Pythagorean identity

$$\cosh^2 x - \sinh^2 x = 1$$

Variants:

$1 - \tanh^2 x = \operatorname{sech}^2 x$.

$\coth^2 x - 1 = \operatorname{cosech}^2 x$.

Addition formulae

$\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B$.

$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$.

$\tanh(A \pm B) = \dfrac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}$.

Double-angle formulae

$\sinh 2A = 2 \sinh A \cosh A$.

$\cosh 2A = \cosh^2 A + \sinh^2 A = 2\cosh^2 A - 1 = 1 + 2\sinh^2 A$.

$\tanh 2A = \dfrac{2 \tanh A}{1 + \tanh^2 A}$.

Derivatives & integrals

$\frac{d}{dx} \sinh x = \cosh x$, $\frac{d}{dx} \cosh x = \sinh x$, $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$.

$\int \cosh x \, dx = \sinh x + C$, $\int \sinh x \, dx = \cosh x + C$.

$\int \operatorname{sech}^2 x \, dx = \tanh x + C$.

Inverse hyperbolic logs

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$ — all x.

$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$ — $x \geq 1$.

$\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$ — $|x| < 1$.

Standard hyperbolic integrals

$\int \dfrac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\dfrac{x}{a} + C$.

$\int \dfrac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\dfrac{x}{a} + C \ (x > a)$.

For $\sqrt{x^2 \pm a^2}$ — use $x = a \sinh u$ or $x = a \cosh u$.

Common pitfalls

- Confusing $\cosh^2 - \sinh^2 = 1$ with the Pythagorean trig identity.
- Forgetting the domain restriction on $\cosh^{-1}$ $(x \geq 1)$.
- Wrong sign in $\frac{d}{dx} \cosh x = \sinh x$ — both positive (unlike $\cos / -\sin$).
- Mixing $\sqrt{x^2+a^2}$ (use sinh) with $\sqrt{a^2-x^2}$ (use sin).

Strategy

1. For equations: convert to e^x, get a quadratic in e^x.
2. For integrals with $\sqrt{x^2+a^2}$: substitute $x = a \sinh u$.
3. For $\sqrt{x^2-a^2}$ with $x > a$: substitute $x = a \cosh u$.
4. Verify log answers using $\sinh^{-1}, \cosh^{-1}, \tanh^{-1}$ formulae.

SECTION T7

Hyperbolic Functions

Questions 1–15 · 122 marks

1. *(a)* Using the exponential definitions of $\sinh x$ and $\cosh x$, show that

$$\cosh 2x = 2\sinh^2 x + 1.$$ [3]

(b) Solve the equation

$$\cosh 2x = 3\sinh x$$

giving your answers correct to three significant figures. [6]

1. *(a)* Use the substitution $x = 2\sinh\theta - 1$ to evaluate the integral

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x^2 + 2x + 5}}.$$

[8]

2. Use the substitution $x = 1 + \sinh\theta$ to evaluate the integral

$$\int_1^2 \sqrt{x^2 - 2x + 2}\ \mathrm{d}x.$$

Give your answer correct to two decimal places. [8]

4. *(a)* Using appropriate definitions in terms of exponential functions, show that

$$\operatorname{sech}^2 x \equiv 1 - \tanh^2 x.$$ [4]

(b) Solve the equation

$$5\operatorname{sech}^2 x = 11 - 13\tanh x$$

giving your answer as a natural logarithm. [8]

1. Solve the equation

$$\cosh 2\theta = 6\sinh\theta - 3.$$

Give your answers in the form $\ln\left(p + \sqrt{q}\right)$, where p, q are positive integers. [7]

3. Use the substitution $x = 2\sinh u$ to evaluate the integral

$$\int_0^2 \frac{dx}{\left(x^2+4\right)^{\frac{3}{2}}} \quad .$$

Give your answer correct to two decimal places. [8]

2. Use the substitution $x = \sinh u$ to evaluate the integral

$$\int_0^3 \frac{x^2}{\sqrt{x^2+1}}\,\mathrm{d}x.$$

Give your answer correct to two decimal places. [7]

1. Find the positive root of the equation

$$3\tanh^2\theta = 5\operatorname{sech}\theta + 1,$$

giving your answer in the form $\ln(a+\sqrt{b})$, where a, b are positive integers. [8]

2. Consider the equation

$$\cosh^2 x = \sinh x + k$$

where k is a constant.

(a) Find the range of values of k for which the equation has no real solution. [4]

(b) Find the positive root of the equation when $k = 3$, giving your answer in the form $\ln(a + \sqrt{b})$, where a, b are positive integers. [3]

1. Determine the two positive roots of the equation

$$\cosh 2x - 7\cosh x + 7 = 0,$$

giving your answers correct to two decimal places. [6]

1. *(a)* Starting with the exponential definition of $\sinh x$, show that

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right).$$ [4]

(b) Solve the equation

$$\cosh 2x = 2\sinh x + 5,$$

giving your answers in the form $\ln\left(a + \sqrt{b}\right)$ where a, b are integers. [5]

7. *(a)* Using the substitution $x = a\sinh\theta$, show that

$$\int \sqrt{x^2 + a^2}\,\mathrm{d}x = \frac{a^2}{2}\left(\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{x^2 + a^2}}{a^2}\right) + \text{constant}.$$ [5]

(b) The equation of the curve *C* is

$$y = x^2, \quad 0 \leqslant x \leqslant 1.$$

Find the arc length of *C*. [6]

END OF PAPER

1. *(a)* Express $5\cosh\theta + 3\sinh\theta$ in the form $r\cosh(\theta + \alpha)$, $r > 0$, where the values of r and α are to be found. [4]

(b) Hence solve the equation

$$5\cosh\theta + 3\sinh\theta = 10.$$ [4]

1. Solve the equation

$$2\sinh\theta + \cosh\theta = 2.$$

Give your answer correct to three significant figures. [7]

$$\cosh^2 x = 3 + \sinh x ,$$

expressing the roots as natural logarithms. [7]

END OF HYPERBOLIC FUNCTIONS PACK

Source: WJEC FP3 (2008 modular spec) · 2005–2017

Curated for WJEC FM 2017 spec A2 Unit 4 — Topic 7 (2.4.7)