Name	Date started	Target end date

GCE A LEVEL — FURTHER PURE MATHEMATICS B QUESTION PACK

0978-01 (Legacy FP2) · New spec A2 Unit 4 Topic 1

REVISE
.wales

FURTHER MATHS — FP B · COMPLEX NUMBERS

Complex Numbers — De Moivre, n-th Roots & the Exponential Form

Every de Moivre / complex-roots question from the legacy WJEC FP2 papers (June 2006 – June 2017 + Specimen) that maps onto the new-spec A2 Unit 4 Topic 1.

LEGACY 2008 SPECIFICATION

Estimated time for entire question pack: ~3 hours 18 minutes

*Derived from the legacy FP2 paper's pace of **~1.5 min/mark** (132 marks over 14 questions). The full Unit 4 exam is **2 hours 30 minutes for 120 marks** (35% of the A-level qualification).*

*You are advised to **not** attempt to complete all of this in one sitting.*

ABOUT THIS QUESTION PACK

This is a **comprehensive practice question pack**, not a single mock paper. It contains every complex numbers question from the legacy WJEC FP2 papers (2008 modular spec) that maps onto new-spec A2 Unit 4 Topic 1 (2.4.1). Unit 4 (Further Pure Mathematics B) is the **120-mark compulsory A2 paper**, worth 35% of the A-level qualification.

Questions are ordered roughly by topic / difficulty.

INSTRUCTIONS

Use black ink or black ball-point pen. Show all working — method marks are awarded for clear setup.

A calculator is allowed. The WJEC Formula Booklet may be referred to.

For Examiner's use only

Q	Source	Max	Mark
1	Jun 07 Q7	8	
2	Jun 08 Q6	8	
3	Jun 08 Q8	10	
4	Jun 10 Q2	9	
5	Jun 10 Q5	8	
6	Jun 11 Q4	12	
7	Jun 12 Q8	14	

Q	Source	Max	Mark
8	Jun 13 Q3	9	
9	Jun 14 Q4	8	
10	Jun 14 Q6	8	
11	Jun 16 Q3	8	
12	Jun 17 Q3	8	
13	Jun 17 Q4	13	
14	Spec. Q5	9	
	Total	**132**	

Complex Numbers — De Moivre, n-th Roots & the Exponential Form — what the new spec asks

WJEC GCE A Level Further Mathematics (from 2017) · Unit 4: Further Pure Mathematics B · Topic 2.4.1.

De Moivre's theorem 2.4.1

- For all real θ and integer n: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
- Proved by induction for positive integers; extended to negative integers using $z^{-n} = 1/z^n$.
- Used to derive multiple-angle formulae for $\cos n\theta$ and $\sin n\theta$.
- For sums of series: $\sum_{k=0}^{n} z^k$ as a geometric series in $z = e^{i\theta}$.

Exponential form & n-th roots 2.4.1

- Definition: $e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's formula).
- Polar form: $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg z$.
- The n distinct n-th roots of $re^{i\theta}$ are $r^{1/n}e^{i(\theta+2k\pi)/n}$ for $k = 0, 1, \ldots, n-1$.
- These roots form the vertices of a regular n-gon centred at the origin.

Multiple-angle formulae 2.4.1

- Expand $(\cos\theta + i\sin\theta)^n$ binomially and equate real / imaginary parts.
- e.g. $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.
- Use $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i\sin n\theta$ to express powers of $\cos\theta, \sin\theta$ in terms of multiple angles.
- Useful for integrals such as $\int \sin^5\theta \, d\theta$.

Working scientifically general

- Always state $|z|$ and $\arg z$ explicitly before taking n-th roots.
- Use the principal argument $-\pi < \arg z \leq \pi$ unless told otherwise.
- For roots, the rotation between successive roots is $2\pi/n$.
- Argand-diagram sketches help you check that the geometry of the answer makes sense (regular polygon).

Complex Numbers in one page

Quick-reference notes — revisit before each section. Don't use during questions.

De Moivre's theorem

For any integer n and real θ:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proved by induction for $n \in \mathbb{Z}^+$.

In exponential form: $(e^{i\theta})^n = e^{in\theta}$.

Exponential form

Polar form using Euler:

$$z = re^{i\theta}$$

where $r = |z|$ and $\theta = \arg z$.

Multiplication: $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$.

Multiple-angle formulae

Let $z = \cos\theta + i\sin\theta$. Then:

$$z^n + z^{-n} = 2\cos n\theta$$

$$z^n - z^{-n} = 2i\sin n\theta$$

Expand $(z + z^{-1})^n$ binomially to express $\cos^n\theta$ in terms of $\cos k\theta$.

n-th roots of a complex number

The n distinct n-th roots of $re^{i\theta}$ are:

$$r^{1/n}e^{i(\theta+2k\pi)/n}$$

for $k = 0, 1, 2, \ldots, n-1$.

They lie on a circle of radius $r^{1/n}$, equally spaced by $2\pi/n$.

Roots of unity

n-th roots of unity: $1, \omega, \omega^2, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i/n}$.

They sum to zero: $\sum_{k=0}^{n-1}\omega^k = 0$ (for $n \geq 2$).

They are vertices of a regular n-gon inscribed in the unit circle.

Sums of trig series

To evaluate $C = \sum\cos k\theta$ and $S = \sum\sin k\theta$:

Form $C + iS = \sum e^{ik\theta}$ (geometric series in $e^{i\theta}$).

Sum: $\dfrac{1 - e^{in\theta}}{1 - e^{i\theta}}$; take real / imaginary parts.

Modulus, argument

$|z|^2 = z\bar{z}$. $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ (mod 2π).

For $z = a + ib$ with $a > 0$: $\arg z = \arctan(b/a)$.

For $a < 0$: add π (or $-\pi$ to stay in principal range).

Common pitfalls

- Forgetting that the n-th roots come in n distinct values — not just one.
- Sign / quadrant error in $\arg z$ — sketch on Argand diagram.
- Using $z^n + z^{-n}$ when problem asks for $z^n - z^{-n}$ — check cos vs sin.
- Stopping at $k = n - 1$ vs $k = n$ — you want n roots, indexed 0 to $n - 1$.

Strategy

1. For roots: convert to polar form first — find r and θ.
2. Apply $r^{1/n}e^{i(\theta+2k\pi)/n}$ for $k = 0, \ldots, n-1$.
3. For multiple-angle: use $z + 1/z = 2\cos\theta$ and binomial expansion.
4. Convert back to $x + iy$ if required.

SECTION T1

Complex Numbers

Questions 1–14 · 132 marks

7. *(a)* Given that

$$z = \cos\theta + i\sin\theta,$$

use de Moivre's Theorem to show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

for all positive integers n. [3]

(b) Hence by expanding $\left(z + \frac{1}{z}\right)^5$, show that

$$\cos^5\theta = a\cos 5\theta + b\cos 3\theta + c\cos\theta$$

where a, b and c are constants to be determined. [5]

6. *(a)* Given that

$$z = \cos\theta + \mathrm{i}\sin\theta,$$

show that

$$z^n - z^{-n} = 2\mathrm{i}\sin n\theta.$$ [3]

(b) Expand $(z - z^{-1})^3$ and hence show that

$$\sin^3\theta = a\sin 3\theta + b\sin\theta$$

where the values of the constants a and b are to be determined. [5]

8. *(a)* Find the modulus and argument of the complex number 8i. [2]

(b) Hence find the three cube roots of 8i, giving your answers in the form $x + \mathrm{i}y$. [8]

2. *(a)* Given that $3 + 4\mathrm{i} = r(\cos\theta + \mathrm{i}\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$, find the values of r and θ. [2]

(b) Hence find the three cube roots of $3 + 4\mathrm{i}$ in the form $x + \mathrm{i}y$. Give the values of x and y correct to three significant figures. [7]

5. Write down de Moivre's Theorem for $n = 5$. Hence show that, for $\sin\theta \neq 0$,

$$\frac{\sin 5\theta}{\sin\theta} = A\cos^4\theta + B\cos^2\theta + C,$$

where A, B, C are constants to be determined.

Deduce the limiting value of $\frac{\sin 5\theta}{\sin\theta}$ as θ tends to zero. [8]

4. Given that $z = -1 + i$,

(a) find the modulus and argument of z, [3]

(b) find the three cube roots of z in the form $x + iy$, giving x and y correct to three decimal places, [7]

(c) find the smallest positive integer n for which z^n is a positive real number. [2]

8. *(a)* Using mathematical induction, prove that

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

for positive integral values of n. [7]

(b) (i) The complex number w is a cube root of the complex number z. Show that $w\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$ is another cube root of z.

(ii) Write down the real cube root of –8. Using the result in (i), or otherwise, find the two complex cube roots of –8, giving your answers in the form $x + \mathrm{i}y$. [7]

3. *(a)* Find the four fourth roots of -1, giving your answers in the form $x + iy$. [6]

(b) (i) Plot the points corresponding to these roots on an Argand diagram.

(ii) The points are joined up to form a square. Find the area of the square. [3]

4. The complex number z is given by $1 + \mathrm{i}\sqrt{3}$.

(a) Find the modulus and the argument of z. [2]

(b) Find the three cube roots of z, giving your answers in the form $x + \mathrm{i}y$ with x and y correct to three decimal places. [6]

6. Using de Moivre's Theorem, show that for $\sin\theta \neq 0$,

$$\frac{\sin 6\theta}{\sin\theta} = a\cos^5\theta + b\cos^3\theta + c\cos\theta,$$

where a, b, c are constants whose values are to be determined.

Hence determine the limiting value of $\dfrac{\sin 6\theta}{\sin\theta}$ as θ tends to π. [8]

3. *(a)* Use de Moivre's Theorem to prove that, for $\sin\theta \neq 0$,

$$\frac{\sin 4\theta}{\sin\theta} = 4\cos\theta\left(1 - 2\sin^2\theta\right).$$ [4]

(b) Hence evaluate

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin 4\theta}{\sin\theta}\,\mathrm{d}\theta.$$

Give your answer correct to three significant figures. [4]

3. Find the three cube roots of the complex number $-8\mathrm{i}$. Give your answers in the form $x + \mathrm{i}y$ where x, y are either integers or surds. [8]

4. *(a)* Given that $z = \cos\theta + \mathrm{i}\sin\theta$, show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

and find a similar expression for $z^n - \frac{1}{z^n}$. [4]

(b) By expanding $\left(z + \frac{1}{z}\right)^5$, show that

$$\cos^5\theta = a\cos 5\theta \; + b\cos 3\theta \; + c\cos\theta\,,$$

where a, b, c are constants whose values should be determined. [5]

(c) Hence evaluate the integral

$$\int_0^{\frac{\pi}{2}} \cos^5\theta \, \mathrm{d}\theta\,.$$ [4]

5. Given that

$$z = \cos\theta + i\sin\theta,$$

use de Moivre's Theorem to show that

$$z^n - \frac{1}{z^n} = 2i\sin n\theta.$$

Hence, by expanding $\left(z - \frac{1}{z}\right)^5$, show that

$$\sin^5\theta = \frac{1}{16}(a\sin 5\theta - b\sin 3\theta + c\sin\theta)$$

where a, b and c are integers to be found. [9]

END OF COMPLEX NUMBERS PACK

Source: WJEC FP2 (2008 modular spec) · 2005–2017
Curated for WJEC FM 2017 spec A2 Unit 4 — Topic 1 (2.4.1)